THE GRAY ABSOLUTE ELECTRODYNAMOMETER.

By Edward B. Rosa.

1. SIMPLE THEORY OF THE INSTRUMENT.

This instrument consists of a cylindrical coil of wire of a single layer suspended within a larger fixed coil, also cylindrical and of a single layer, the ratio of the radius of each coil to its length being as one to the square root of 3. When these two coils have this particular shape and their centers coincide, the expression for the torque which in general is given by a series of terms is simplified by the disappearance of all terms after the first up to the seventh, and this and succeeding terms are usually small enough to be neglected.[1] The torque is then expressed, if the axes are at right angles, as is usually the case in practice, by the following extremely simple formula:

$$T=\frac{2\pi^2 r^2 N_1 N_2 I_1 I_2}{c} \tag{1}$$

where r is the radius of the suspended coil.

N_1 and N_2 are the whole number of turns of wire on the fixed and suspended coils respectively,

I_1 and I_2 are the currents in the two coils respectively, which may or may not be the same current,

c is half the diagonal of the fixed coil $=\sqrt{a^2+b^2}$, a being the radius of the fixed coil and $2\,b$ its length.

As stated above $\frac{a}{2b}=\frac{1}{\sqrt{3}}$.

[1] Gray, Absolute Measurements, Vol. II, part I, p. 276. Patterson, Physical Review, **20**; 1905.

The seventh and succeeding terms, which amount only to extremely small correction terms, are appreciable only when the suspended coil is relatively large; they can readily be calculated if necessary.

The formula may also be written as follows:

$$\left.\begin{aligned} T &= \frac{2\pi N_1 I_1}{c} \cdot \pi r^2 N_2 I_2 \\ &= HI_1 \times AI_2 \end{aligned}\right\} \qquad (2)$$

where $H=\frac{2\pi N_1}{c}$, and $A=\pi r^2 N_2$. H is the magnetic force at the center of the fixed coil, due to unit current flowing through the fixed coil, and A is the sum of the areas of the several turns of the suspended coil. Thus the torque is the same as though the suspended coil were hung in a uniform field of strength $H_1=HI_1$. The field is, of course, not uniform, the center O being a maximum point with respect to the axis A B of the fixed coil (Fig. 1), and a minimum point with respect to the axis of the suspended coil, which is a diameter of the fixed coil. This saddle-shaped field gives a torque, however, exactly the same (within 1/27,000 when $r=1/2\ a$ and still closer for smaller values of r such as are used in practice), as though the field were uniform and of the same intensity as it has at the center, O.

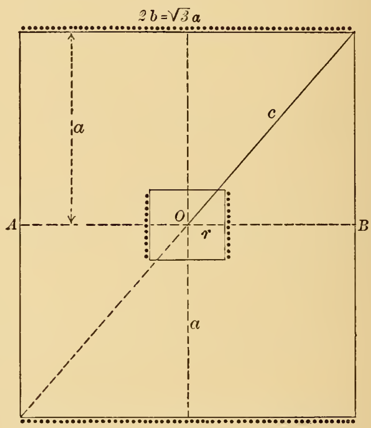

Fig. 1.

This form of electrodynamometer has been used by Patterson and Guthe[1] and Carhart and Guthe[2] in the absolute measurement of current and of the electromotive force of a standard cell, and has been used by Guthe in a recent very careful determination at the Bureau of Standards, an account of which is published in the present number of this Bulletin. The instrument is extremely well

[1] Physical Review, 7; 1898. [2] Physical Review, 9; 1899.

adapted to precision measurements, the quantity most difficult to determine with sufficient accuracy being the torque. Guthe and others have, however, succeeded so well in this respect that it seems worth while to construct yet another instrument of the same style, and to refine the construction and measurements still further if possible, with the hope of fixing yet more definitely the value of the standard cell.

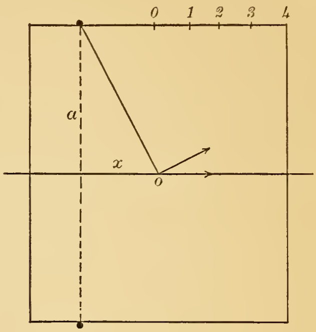

Fig. 2.

I propose to examine the theory of the instrument with respect to sources of error arising in its construction and measurement, leaving aside for the present the question of errors arising in carrying out the experiment.

The formula (2) expresses the torque in terms of the field H at the center of the fixed coil. The force at O in the direction of the axis due to unit current in a single turn of wire is

$$H_1 = 2\pi \frac{a^2}{(a^2+x^2)^{\frac{3}{2}}} \tag{3}$$

To find the force due to a current sheet of length 2 b with a current of n units per centimeter, we integrate along the length of the current sheet, thus

$$H = 2\pi a^2 \int \frac{n dx}{(a^2+x^2)^{\frac{3}{2}}} = 2\pi n \left[\frac{x}{\sqrt{a^2+x^2}}\right]_{-b}^{+b} = 2\pi n \frac{2b}{\sqrt{a^2+b^2}} \tag{4}$$

If the current sheet is replaced by a winding of fine wire having n turns per centimeter and carrying unit current, there being altogether $N_1 = 2bn$ turns, we have

$$H = \frac{2\pi N_1}{\sqrt{a^2+b^2}} = \frac{2\pi N_1}{c} \text{ as above, (2),}$$

provided the single layer winding of wire has the same magnetic effect at the center that the current sheet has. In other words, the formula given above for the Gray dynamometer applies strictly to the case of a current sheet on each of the two cylindrical coils, and

we have to inquire whether a winding of insulated wire is exactly equivalent to the current sheet, and how to measure the coils in getting a, c, and r for the formula.

2. A SINGLE LAYER OF WIRE EQUIVALENT TO A CURRENT SHEET. LENGTH OF THE COILS.

Considering first the case of the fixed coil, it is evident that if it be wound with flat strip of negligible thickness and of width equal to the outside diameter of the insulated wire, and the same number of turns are applied as there are of wire, then the edges of the strip will meet (without making contact, it is supposed) and the current flowing through the strip will envelop the cylinder like a current sheet. If there are n turns per centimeter, unit current in the strip will have the same magnetic effect at the center of the coil as a current of n units per centimeter, in the current sheet. We may therefore consider the formula exact for the assumed winding of the strip, and consider whether the winding of insulated wire is magnetically equivalent to the strip. In the fixed coil of the electrodynamometer used by Guthe there are about 20 turns per centimeter. I shall therefore consider the case of wire .05 cm in diameter. Equation (4) gives the force at the center due to any portion of a current sheet, and if we insert for the limits the values of x at the two edges of a single turn of the strip we shall have the force due to one turn of strip. Thus, since $n=20$ and $a=25$ cm, we have for a single turn at the center of the cylinder, the values of x for the edges being $+.025$ and $-.025$,

$$\mathrm{H}_1=2\pi\times 20\left[\frac{x}{\sqrt{a^2+x^2}}\right]_{-.025}^{+.025}=\frac{2\pi}{a}\left[\frac{0.5}{\sqrt{1+(.001)^2}}+\frac{0.5}{\sqrt{1+(.001)^2}}\right] \quad (5)$$

$$\text{or } \mathrm{H}_1=\frac{2\pi}{a}\times .9999995$$

A single turn of infinitely fine wire at the center ($x=0$) would give a force H at the center O equal to

$$H_2=2\pi\frac{a^2}{(a^2)^{\frac{3}{2}}}=\frac{2\pi}{a}$$

The difference H_2-H_1 is only one part in 2,000,000 of the force, and hence the single filament is practically equivalent to the strip.

If the strip were 1 cm wide the difference would be 400 times as great, or one part in 5,000. If the strip is wound on edge and carries unit current the force due to an element dy of one turn at a distance $a+y$ (where a is measured to the center of the strip) is

$$d\,H_1=\frac{2\pi}{a+y}\cdot\frac{dy}{2\alpha}\cdot\frac{a}{a+y}$$

where 2α is the depth of the strip, and $\frac{a}{a+y}$ is the relative current density at the element dy as compared with its value at the center of the strip. This is not strictly the mean value, but it is nearly enough true for our purpose when the radius is large. The whole force is then

$$H_1=\frac{2\pi a}{2\alpha}\int_{-\alpha}^{+\alpha}\frac{dy}{(a+y)^2}=\frac{2\pi a}{2\alpha}\left[-\frac{1}{a+y}\right]_{-\alpha}^{+\alpha} \qquad (6)$$

$$=\frac{2\pi}{a}\left[\frac{a^2}{a^2-\alpha^2}\right]$$

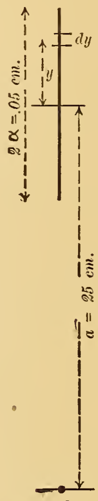

Fig. 3.

If as before, $a=25$ and $\alpha=.025$,

$$H_1=\frac{2\pi}{a}\Big[1.000001\Big]$$

$$\text{whereas } H_2=\frac{2\pi}{a}$$

$$\text{Thus } H_1-H_2=\frac{2\pi}{a}\times 0.000001$$

That is, the force due to unit current in the strip on edge exceeds that due to unit current in a filament at its center by one part in a million. If the strip were 1 cm deep the difference would be appreciable but small.

It is thus evident that for a wire of small diameter at the center of the cylinder the effect of the breadth of the wire is to make the force at the center slightly less, and the effect of the depth is to make it slightly greater, than if the current were concentrated at the center of the cross section of the wire; hence the resultant difference between the effect of unit current in a round wire .05 cm in diameter and a filament at its center is even less than that between the strip

and the filament, and hence the wire may be considered as equivalent to the strip to within less than one in a million.

If the strip is near the end of the cylinder where $x=20$, say, its force at the center is found similarly from equation (4). Thus,

$$H_1 = 2\pi \times 20 \left[\frac{x}{\sqrt{a^2+x^2}}\right]_{20-a}^{20+a} \tag{7}$$

$$= 2\pi \times 20 \left[\frac{20.025}{\sqrt{625+(20.025)^2}} - \frac{19.975}{\sqrt{625+(19.975)^2}}\right] = .11966704$$

whereas the force due to a filament at the center of the strip is

$$H_2 = 2\pi \frac{625}{(625+400)^{\frac{3}{2}}} = .11966695$$

Thus, H_1 is larger than H_2 by about one part in a million, whereas at the center of the cylinder it is smaller by one part in 2,000,000. At a distance of 12.5 cm from the center the two are exactly equal. Thus the difference between the winding of fine insulated wire of 20 turns per cm and thin strip which is exactly equivalent to a current sheet is probably less than one part in 5,000,000. If the strip is 1 cm wide instead of one-twentieth of a centimeter, the difference at the end of the cylinder is only one part in 8,600, the effect of the strip being greater, whereas at the center of the cylinder the difference, as we saw above, was one part in 5,000, the effect of the strip being less. Thus, the average difference will probably be less than one in 12,500 making an error in the current of only one in 25,000. In other words, if a winding of strip 1 cm wide for which the formula of the electrodynamometer applies were replaced by a winding of fine wire 1 cm apart, the error would scarcely be appreciable. It is to be noted that the length of the cylinder to be used in the formula would be the total length of the equivalent current sheet, and not the length of the winding of wire. Thus if the cylinder were 43 cm long, the 43 turns of strip would be replaced by 43 turns of fine wire. The distance between the first and last wires would be 42 cm; but the length of the cylinder would be taken as 43 cm, and not 42, in calculating the force at the center. This is an extreme case, but the same principle applies always, namely, that the length of the

cylinder is to be the length of the equivalent current sheet, and in a winding of insulated wire this is the *overall length of the winding*, AB, *including the insulation on the first and last wires.* It is generally supposed that the measurement should be made a little short of the outside dimensions as *ab*, not including the insulation. The difference is not large in a winding of fine wire, but it is appreciable even in that case; amounting to about two parts in 10,000 in a winding of 20 turns per centimeter, with a radius of 25 cm.

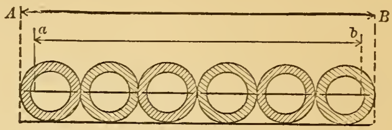

Fig. 4.

3. EFFECT OF IRREGULAR WINDING.

Equation (3) gives the effect at the center of unit current in a single turn at distance x from the center. Differentiating this with respect to x we have

$$dH = -2\pi a^2 \frac{3x dx}{(a^2+x^2)^{\frac{5}{2}}} \tag{8}$$

which gives the effect of displacing the wire parallel to itself by a distance dx. Dividing (8) by (3) we have for the relative change in the magnetic force due to a displacement dx,

$$\frac{dH}{H} = -\frac{3x dx}{a^2+x^2}$$

Thus the relative change is proportional to x, the distance from the central plane and inversely proportional to the square of the distance of the wire from the center of the cylinder O. To find the actual value of this change we may substitute in (9) numerical values for a and x. Putting $a = 25$ cm and assuming a displacement dx of only 0.1 mm, and giving x values corresponding to the points 0, 1, 2, 3, 4, in Fig. 2 (that is at the center, and at quarter, half, and three-quarters of the distance from the center to the end, and finally at the end), we have

$$\text{At point 4, } x = \tfrac{1}{2}\sqrt{3}\, a,\ \frac{dH}{H} = \frac{1}{1700}$$

$$\text{At point 3, } x=\tfrac{3}{8}\sqrt{3}\,a,\ \frac{dH}{H}=\frac{1}{1820}$$

$$\text{At point 2, } x=\tfrac{1}{4}\sqrt{3}\,a,\ \frac{dH}{H}=\frac{1}{2300}$$

$$\text{At point 1, } x=\tfrac{1}{8}\sqrt{3}\,a,\ \frac{dH}{H}=\frac{1}{4000}$$

$$\text{At point 0, } x=0 \qquad \frac{dH}{H}=\frac{1}{\infty}$$

Thus we see that the effect of displacing a given wire by so little as 0.1 mm produces an appreciable effect, and if a large number of wires are so displaced, due to irregular winding, the effect may be serious if it is not calculated and a correction applied. Of course, in any irregular winding the displacements will be both plus and minus, and hence the resultant effect will be the difference due to the two kinds of displacement. The effect is, however, so large that this difference may be very important in precise work.

A good approximation to a correction for the effect of irregular winding may be made by dividing the winding into a number of sections and calculating the magnetic effect at the center due to each section by equation (4) using for the limits the values of x at the edges of the section. This requires a careful calibration of the coil, such as Guthe has made in his recent work. He measured the length of each 50 turns of the winding at several different equidistant elements of the cylinder, and took the averages as the breadth of each section. There being 872 turns in all, there are 17 sections of 50 turns each and 1 section of 22 turns. The breadths of these sections are given in column 3 of Table 1. Column 4 gives the distance from one end of the successive boundaries, found by adding successively the numbers in column 3. Column 5 gives the distance from the center of each boundary, found by subtracting the half length b from the numbers of column 4. These are the values of x to be used in equation (4) in calculating the successive values of H at the center. Column 6 gives the corresponding values of $\frac{H}{2\pi}$.

The magnetic force due to unit current in the first section is found as follows:

$$H_1 = 2\pi n_1 \left[\frac{x}{\sqrt{a^2 + x^2}} \right]_{x_1}^{x_2} \tag{10}$$

where n_1 is the number of turns per centimeter, and x_1 and x_2 are the distances of the edges of the section of n turns from the center. Since $x_1 - x_2$ is the breadth of the section, $n_1 = \frac{n}{x_1 - x_2}$. Hence, equation (10) becomes

$$H_1 = \frac{2\pi n}{x_1 - x_2} \left[\frac{x_1}{\sqrt{a^2 + x_1^2}} - \frac{x_2}{\sqrt{a^2 + x_2^2}} \right] \tag{11}$$

Substituting $n = 50$,
$x_1 = 21.6382$,
$x_2 = 19.1437$,
$a = 24.9812$, the mean radius of the windings, we find that $H_1 = 2\pi \times 0.931245$. The same operation being carried out for each section gives the values contained in the last column of Table I, from which it follows that the magnetic force at the center of the coil is $H = 165.992$.

If we assume the winding uniform we should have, since c the half diagonal $= \sqrt{a^2 + b^2} = \sqrt{1092.272}$,

$$H = 2\pi \frac{872}{\sqrt{1092.272}} = 165.778.$$

This value of H, the magnetic force at the center of the cylinder, is too small by 0.214, or 1 in 800; and this would cause an error of 1 in 1,600 in the value of the current found. It is thus seen that the error which would arise from assuming the winding uniform in this case amounts to more than the uncertainty in the absolute value of the standard cell. As this coil was carefully wound, and its winding looks very smooth and uniform, there is no reason to suppose it is worse than such coils often are. Hence, we may conclude that any value obtained from an absolute dynamometer of this type in which uniformity of winding is not assured, or in which the effect of irregularity in the winding is not taken into

account, is untrustworthy. It is of course desirable that the winding be put on as uniformly as possible, so that the calculated correction may be quite small.

TABLE I.

Showing the Breadth and Location of Each Section of Winding of Guthe's Electrodynamometer, and Its Magnetic Force at the Center.

Number of section	Number of turns	Breadth of section	Distance from end	Distance from center $=x$	Values of $H \div 2\pi$ for each section
		cm	0.0000	−21.6382	
1	50	2.4945	2.4945	−19.1437	0.931245
2	50	2.4940	4.9885	−16.6497	1.075830
3	50	2.4970	7.4855	−14.1527	1.234935
4	50	2.4887	9.9742	−11.6640	1.403505
5	50	2.4887	12.4629	− 9.1753	1.573075
6	50	2.4758	14.9387	− 6.6995	1.731570
7	50	2.4652	17.4039	− 4.2343	1.864215
8	50	2.4709	19.8748	− 1.7634	1.956820
9	50	2.4696	22.3444	+ 0.7062	1.997730
10	50	2.4639	24.8083	+ 3.1701	1.981245
11	50	2.4575	27.2658	+ 5.6276	1.909995
12	50	2.4588	29.7246	+ 8.0864	1.793570
13	50	2.4766	32.2012	+10.5630	1.644875
14	50	2.4804	34.6816	+13.0434	1.479280
15	50	2.4978	37.1794	+15.5412	1.309140
16	50	2.4953	39.6747	+18.0365	1.144890
17	50	2.4946	42.1693	+20.5311	0.993455
18	22	1.1071	43.2764	+21.6382	0.393080
		43.2764		Total..	26.418455

$$H = 2\pi \times 26.418455 = 165.992$$

The winding of wire being spiral, it is evident that the distance from the central plane to the end turn is different at different elements of the cylinder. One half a turn will be farther and the other half nearer than the mean distance, the average displacement being one-fourth of the diameter of the wire. The resultant will be the difference of the effects of these two opposite displacements, and

this compared with the total force at the center will be extremely small. Substituting the numerical values in equation (9) it can readily be shown that the variation due to the spiral winding from that calculated on the supposition that each turn is in a plane perpendicular to the axis is not as much as one part in two million.

4. EFFECT OF AN ERROR IN THE RADIUS OR LENGTH OF THE FIXED COIL. TEMPERATURE COEFFICIENTS.

To find the effect on the computed value of the magnetic field due to a small error in measuring the radius we differentiate the expression for the total force at the center.

Since

$$H = \frac{2\pi N}{(a^2+b^2)^{\frac{1}{2}}},$$

$$dH = -2\pi N \frac{a\,da}{(a^2+b^2)^{\frac{3}{2}}}$$

$$\frac{dH}{H} = -\frac{a\,da}{a^2+b^2} = -\frac{a^2}{a^2+b^2}\cdot\frac{da}{a} = -\frac{4}{7}\frac{da}{a} \qquad (12)$$

since $b^2 = \frac{3}{4}a^2$. Similarly,

$$\frac{dH}{H} = -\frac{b\,db}{a^2+b^2} = -\frac{b^2}{a^2+b^2}\cdot\frac{db}{b} = -\frac{3}{7}\frac{db}{b} \qquad (13)$$

Thus, the relative error in the magnetic field is four-sevenths of the relative error in the radius, and three-sevenths of the relative error in the measurement of the length. If an error of 0.1 mm is made in determining the mean diameter, which is 50 cm, that is one part in 5,000; the error in the computed magnetic field will be 1 part in 8,750. If an error of 0.1 mm be made in measuring the length of the winding, which is 43.3 cm, that is one part in 4,330; the error in the computed magnetic field will be 1 in 10,100.

If a and b change together due to change of temperature, the relative change in the magnetic field due to unit current will be the sum of the effects due to change in a and b, or

$$\frac{dH}{H} = -\left(\frac{3}{7}+\frac{4}{7}\right)\frac{da}{a} = -\frac{da}{a} \qquad (14)$$

That is, the temperature coefficient for the magnetic field is negative, and equal numerically to the temperature coefficient for the material on which the coil is wound. If the cylinder is marble, this is 1 part in 100,000 per degree Centigrade. For plaster of Paris Guthe found it to be 1 part in 40,000 per degree.

This expression for the temperature coefficient may be directly deduced as follows. If the coil expands equally in all directions, thus maintaining the same form, $b^2=\frac{3}{4}a^2$ always, and

$$H=\frac{2\pi N}{\left(a^2+\frac{3}{4}a^2\right)^{\frac{1}{2}}}=\frac{2\pi N}{\left(\frac{7}{4}\right)^{\frac{1}{2}}a}$$

$$dH=-\frac{2\pi N}{\left(\frac{7}{4}\right)^{\frac{1}{2}}}\cdot\frac{da}{a^2}$$

$$\therefore \frac{dH}{H}=-\frac{da}{a}, \text{ as above.}$$

5. EFFECT OF THE OPENING IN THE COIL FOR THE SUSPENSION.

An opening is left in the winding of the fixed coil through which the suspension passes. Several wires near the middle plane are carried around the oblong hard-rubber bushing which lines this hole, the length of the bushing being perhaps 2 cm. To calculate the effect of this irregularity in the winding we may consider the displacement laterally of four wires through a distance of 2 mm on each side of the central plane, and a simultaneous radial displacement of the same wires through half a millimeter, as they are laid upon the adjacent wires. The length of wire so displaced may be taken as 3 cm. These dimensions are greater than those of Guthe's dynamometer, and will give an outside value for the correction.

Equation (9) gives the relative change in the magnetic field at the center due to a displacement of any turn of wire parallel to itself through a distance dx. Integrating it with respect to x we have the relative change as a wire is displaced through a (small) finite distance x_2-x_1. For a displacement from $x_1=0$ to $x_2=0.2$ cm we thus have ($a=25$ cm),

$$\int_0^{0.2}\frac{3xdx}{a^2+x^2}=\frac{3}{2}\log\frac{a^2+0.04}{a^2}=\frac{3}{2}\cdot\frac{0.04}{625}=\frac{1}{10,000}\text{ approximately.}$$

Displacing the other three wires from $x_1=.05$ to $x_2=.25$, $x_1=.10$ to $x_2=.30$, $x_1=.15$ to $x_2=.35$ respectively, produces a slightly greater change, the mean for the four being 1 part in 6,000. This supposes the wires displaced for the whole length of each turn, whereas the displacement is only for 3 cm out of 157. Hence the total relative change is only one part in 300,000 of the eight central wires, or one part in about one hundred and fifty million of the whole force at the center.

Similarly, by the use of equation (12) we can estimate the effect of half a millimeter radial displacement of the same eight wires, four on each side of the central plane. For $da=.05$ cm, $\frac{da}{a}=.002$. Hence for 3 cm of the 157 in one circumference we have

$$\frac{dH}{H}=\frac{4}{7}\times.002\times\frac{3}{157}=0.000022.$$

But these eight wires exert only about one-fiftieth of the total force at the center, hence the relative change in the total force due to the radial displacement is less than one in two million. Hence we see that the change in the field due to the irregularity in the winding occasioned by this opening is so small that it is not wise to cramp the experiment by making it unduly small.

6. EFFECT OF ERROR IN MEASURING R, AND EFFECT OF THICKNESS OF THE WIRE OR STRIP.

Since $A=\pi r^2N$, the torque is proportional to the square of the radius. Thus

$$\frac{dT}{T}=2\frac{dr}{r} \tag{15}$$

That is, the relative error in the computed torque is twice the relative error in measuring the radius of the small coil. It is therefore important to measure this radius with great exactness, and one

should use a moving coil large enough to permit this measurement to be made with the necessary accuracy. This error includes, of course, not simply the error in measuring the cylinder, but of getting the exterior diameter after winding. Flat strip is better than round wire on this account. The temperature coefficient of the torque, due to expansion or contraction of the suspended coil, is, of course, twice the temperature coefficient of the material of which the suspended cylinder is made. It is usual to take the radius of the coil as the distance from the center of the cylinder to the center of the wire or strip; or, what is the same thing, the mean of the radius of the bare cylinder and of the cylinder after it is wound. This is not quite exact, whether we assume the current to be uniformly distributed through the cross section of the wire, or assume it to be inversely proportional to the distance from the center, but for thin strip the error is inappreciable in either case, as the following demonstration shows. The torque is

$$T = CI_1 \pi r_0^2 N_2 I_2,$$

where r_0 is the equivalent radius of the suspended coil and I_2 is the total current flowing in it.

First, assume that the current density at any point in the strip is inversely proportional to its distance r from the axis of the cylinder, then

$$di = a\frac{dr}{r}$$

The whole current I in the strip is

$$I = a\int_{r_1}^{r_2}\frac{dr}{r} = a\log\frac{r_2}{r_1}, \therefore a = \frac{I}{\log\frac{r_2}{r_1}}.$$

If r_0 is the equivalent radius of the strip, that is, the radius of the equivalent current sheet,

$$r_0^2 I = a\int_{r_1}^{r_2} r^2 di = \frac{I}{\log\frac{r_2}{r_1}}\int_{r_1}^{r_2} r dr = \frac{I}{\log\frac{r_2}{r_1}}\cdot\frac{r_2^2 - r_1^2}{2}$$

or, $$r_0^2 = \tfrac{1}{2}\cdot\frac{r_2^2 - r_1^2}{\log\dfrac{r_2}{r_1}} \tag{16}$$

This expression may be put into more convenient form for calculation as follows, expanding the $\log\dfrac{r_2}{r_1}$,

$$r_0^2 = \frac{r_2+r_1}{2}\left(\frac{r_2-r_1}{\dfrac{r_2-r_1}{r_1} - \dfrac{1}{2}\left(\dfrac{r_2-r_1}{r_1}\right)^2 + \dfrac{1}{3}\left(\dfrac{r_2-r_1}{r_1}\right)^3 - \dfrac{1}{4}\left(\dfrac{r_2-r_1}{r_1}\right)^4 + \ldots}\right)$$

$$= \frac{r_2+r_1}{2}\left(\frac{r_1}{1 - \dfrac{1}{2}\left(\dfrac{r_2-r_1}{r_1}\right) + \dfrac{1}{3}\left(\dfrac{r_2-r_1}{r_1}\right)^2 - \dfrac{1}{4}\left(\dfrac{r_2-r_1}{r_1}\right)^3 + \ldots}\right)$$

$$= \left(\frac{r_2+r_1}{2}\right) r_1 \left\{1 + \frac{r_2-r_1}{2r_1} - \frac{1}{12}\left(\frac{r_2-r_1}{r_1}\right)^2 + \frac{1}{24}\left(\frac{r_2-r_1}{r_1}\right)^3 + \ldots\right\}$$

or, finally,

$$r_0^2 = \left(\frac{r_2+r_1}{2}\right)^2 \left\{1 - \frac{1}{6}\,\frac{(r_2-r_1)^2}{r_1(r_1+r_2)} + \frac{1}{12}\,\frac{(r_2-r_1)^3}{r_1^2(r_1+r_2)} - \ldots\right\} \tag{17}$$

Thus the equivalent radius r_0 is less than the mean radius $\dfrac{r_2+r_1}{2}$, the difference being chiefly due to the first of the correction terms in the above expression. If $r_1 = 5$ cm, $r_2 - r_1 = .05$ cm, the quantity in brackets above becomes

$$1 - 0.000008292 + 0.000000042 - \ldots\ldots$$

Thus the correction to be applied to $\left(\dfrac{r_1+r_2}{2}\right)^2$ to give the true value of r_0^2 amounts to one part in about 121,000 and is *negative.*

Second, suppose the current density is uniform over the cross section of the strip, then

$$di = \frac{I}{r_2 - r_1}\,dr$$

$$r_0^2 I = \frac{I}{r_2-r_1}\int_{r_1}^{r_2} r^2\,dr = \frac{I}{r_2-r_1}\left[\frac{r_2^3 - r_1^3}{3}\right]$$

$$\therefore\ r_0^2 = \frac{1}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right) = \left(\frac{r_2+r_1}{2}\right)^2 \left\{1 + \frac{1}{3}\left(\frac{r_2-r_1}{r_2+r_1}\right)^2\right\} \tag{18}$$

Thus the equivalent radius r_0 is again expressed in terms of the mean radius, there being a single correction term which is almost identical with the principal correction term in (17), but is *positive.* Using the same numerical values as before we find the correction amounts to one part in 121,203. Thus the mean radius $\frac{r_2+r_1}{2}$ is too large by the first hypothesis as to the distribution of current and too small by the second, the error in either case being inappreciable if the thickness of the strip is not more than one per cent of the radius. As we can not be sure of the exact distribution of current, owing to lack of perfect homogeneity in the wire, the better way is to use strip thin enough to make small variations in the distribution unimportant.

We have assumed throughout the foregoing discussion that the changes in the torque due to any irregularity of winding or other departure from the conditions for which the simple formula holds are proportional to the change produced in the field at the center of the cylinder. This is justifiable, since such changes are assumed to be small, and variations could at most be only small quantities of the second order.

The above examination shows that it is of first importance to secure a uniform winding of the fixed coil, and to apply a correction for any unavoidable irregularity that may be found by careful calibration to remain. It also gives the length of the current sheet equivalent to a winding of insulated wire, indicates the order of magnitude of certain possible sources of error, and suggests how to render them inappreciable. These principles will be applied in the new absolute dynamometer, in which a marble cylinder will be used for the fixed coil, shortly to be constructed at the Bureau of Standards.